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On p-adic Hodge theory for semi-stable families

Takeshi Tsuji

Research Institute for Mathematical Sciences, Kyoto University
Kyoto, 606-01, Japan

Let K be a complete discrete valutation ring of characteristic 0 with perfect residue field k of characteristic p and let O_K be its ring of integers. Let X be a proper semi-stable scheme over O_K. Under the restriction dim $X_K < (p-1)/2$, K. Kato has proved in [Ka3] a conjecture of Fontaine-Jannsen (Conjecture 1.3 below) which compares the p-adic etale cohomology $H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ with the log. crystalline cohomology $H^m_{\text{st}}(X)$. Here we remove the restriction on the dimension when $p \geq 3$. In his proof, the restriction on the dimension arose from the fact that he proved the isomorphism between p-adic etale cohomology and syntomic cohomology (with log. poles) only when the degree of the cohomologies is smaller than $p-1$. Hence, in order to remove the restriction, we only have to prove this isomorphism without the restriction on the degree. (See Theorem 1.1.)

In this note, we first state the results in §1, and then, in §2–§4, give a sketch of the proof of the main theorem (Theorem 1.1) in the case where X is a usual proper smooth scheme over O_K. See [Tsu] for details.

1. Result

Let K, k and O_K be as above, and let X be a semi-stable scheme over O_K. Here semi-stable means that X_K is smooth over K, X is regular, and the special fiber $Y := X \otimes k$ is a reduced divisor with normal crossings in X.

Let (S, N) be the scheme Spec O_K with the log. str. defined by its closed point, and let M be the log. str. on X defined by its special fiber Y. (In this note, the log. str. always means the one defined by Fontaine-Illusie. See [Ka2].) Then the canonical morphism $(X, M) \to (S, N)$ is smooth.

We define the syntomic cohomology (with log. poles) $H^m(\overline{X}, S_{\mathbb{Q}_p}(r)_{\overline{X}})$ as follows. First assume that we are given globally a closed immersion $i\colon (X, M) \hookrightarrow (Z, M_Z)$ with (Z, M_Z) log. smooth over W and a compatible system of liftings of frobenius $\{F_{Z_n}\colon (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\}$. Here W is a ring of Witt vectors with coefficients in k, and the subscript n denotes the mod p^n reduction. Describe by (D_n, M_{D_n}) the PD-envelope of $i \otimes \mathbb{Z}/p^n\mathbb{Z}$ and by J_{D_n} the PD-ideal of $\mathcal{O}_{D_n}$. Define the complex $\mathcal{S}_n^{\sim}(r)_{X,Z}$ on Y_{et} to be the mapping fiber of the morphism of complexes

$$p^r - \varphi\colon J_{D_n}^{[r-\cdot]} \otimes \Omega^{\cdot}_{Z_n/W_n}(\log M_{Z_n}) \longrightarrow \mathcal{O}_{D_n} \otimes \Omega^{\cdot}_{Z_n/W_n}(\log M_{Z_n}),$$

where φ denotes the morphism induced by F_{Z_n}. Up to canonical quasi-isomorphisms, this complex is independent of the choice of i and $\{F_{Z_n}\}$.

For a general X, we define

$$\mathcal{S}_n^{\sim}(r)_X \in D^+(Y_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})$$

by "gluing" $\mathcal{S}_n^{\sim}(r)_{X,Z}$, and

$$\mathcal{S}_n^{\sim}(r)_{\overline{X}} \in D^+(\overline{Y}_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})$$

by taking "inductive limit" of $\mathcal{S}_n^{\sim}(r)$ for $(X, M) \times_{(S,N)} (\text{Spec } O_{K'}, N')$, where K' runs through all finite sub-extensions of K in an algebraic closure $\overline{K}$ of K and N' is the log. str. defined by the closed point.

Finally we define the cohomology $H^m(\overline{X}, S_{\mathbb{Q}_p}(r)_{\overline{X}})$ with canonical action of the Galois group $G_K := \text{Gal}(\overline{K}/K)$ by

$$(\varprojlim_n H^m_{\text{et}}(\overline{Y}, \mathcal{S}_n^{\sim}(r)_{\overline{X}})) \otimes \mathbb{Q}_p.$$

In the following, we assume that X is proper over O_K.

Theorem 1.1. *If $p \geq 3$, we have a canonical Galois equivariant isomorphism*

$$H^m(\overline{X}, S_{\mathbb{Q}_p}(r)_{\overline{X}}) \cong H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p(r))$$

for $r \geq m \geq 0$.

Remark 1.2. This has been already proved by K. Kato in [Ka3] if $r < p - 1$.

Let $H^m_{\text{st}}(X)$ be the Hyodo-Kato cohomology of (X, M), which is a K_0-vector space ([H2], [HK]). Here K_0 is a field of fractions of W. As additional structures, it has a semi-linear automorphism φ called "frobenius" and a nilpotent endomorphism N called "monodromy operator" which satisfy the relation;

$$p\varphi N = N\varphi.$$

O. Hyodo and K. Kato proved that $H^m_{\text{st}}(X) \otimes_{K_0} K$ is isomorphic to $H^m_{\text{dR}}(X_K/K)$, and hence it admits a Hodge filtration. Let B_{st} be the ring defined by J.-M. Fontaine [Fo3], with the action of the Galois group G_K, the frobenius φ, the monodromy operator N and the filtration $Fil^{\cdot}$ after $\otimes_{K_0} K$.

Conjecture 1.3 (Fontaine-Jannsen). *There is a canonical isomorphism*

$$B_{\text{st}} \otimes_{K_0} H^m_{\text{st}}(X) \cong B_{\text{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$$

compatible with the actions of Galois group G_K, φ, N, and the filtrations after $\otimes_{K_0} K$.

Remark 1.4. By this isomorphism, these two cohomologies with their additional structures can be recovered from each other in the following manner;

$$H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \cong Fil^0(B_{\text{st}} \otimes_{K_0} H^m_{\text{st}}(X))^{N=0, \varphi=1}$$
$$H^m_{\text{st}}(X) \cong (B_{\text{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K}.$$

By the argument of K. Kato, we obtain the following theorem from Theorem 1.1.

Theorem 1.5. *The conjecture of Fontaine-Jannsen is true if $p \geq 3$.*

Remark 1.6. This has been proved by K. Kato in [Ka3] if $\dim X_K < (p-1)/2$.

Next we consider the open case. Let X be as above and let D be a reduced divisor with normal crossings on X which satisfies the following two conditions:

(1) If $D = \sum_i D_i$ with D_i a prime divisor for each i, the scheme D_i with the inverse image log. str. of M is log. smooth over (S, N) for each i.

(2) Etale locally on X, there is an etale morphism

$$X \longrightarrow \operatorname{Spec} O_K[T_1, \cdots, T_r, S_1, \cdots, S_s]/(T_1 \cdots T_r - \pi)$$

such that $D = \{S_1 \cdots S_s = 0\}$.

Then we can define syntomic cohomology

$$H^m(\overline{X}, S_{\mathbb{Q}_p}(r)(\log D)_{\overline{X}}) \quad (\text{resp. } H^m(\overline{X}, S_{\mathbb{Q}_p}(r)(-\log D)_{\overline{X}})$$

by adding "log. poles along D" to $\mathcal{S}_n^{\sim}(r)_{\overline{X}}$ (resp. by tensoring "the ideal defining D" to $\mathcal{S}_n^{\sim}(r)(\log D)_{\overline{X}}$). Let U be the complement of D in X.

Theorem 1.7. *If $p \geq 3$, there are canonical Galois equivariant isomorphisms*

$$H^m(\overline{X}, S_{\mathbb{Q}_p}(r)(\log D)_{\overline{X}}) \cong H^m_{\text{et}}(U_{\overline{K}}, \mathbb{Q}_p(r))$$
$$H^m(\overline{X}, S_{\mathbb{Q}_p}(r)(-\log D)_{\overline{X}}) \cong H^m_{\text{et},\, c}(U_{\overline{K}}, \mathbb{Q}_p(r))$$

for $r \geq m \geq 0$.

2. Preliminaries on A_{crys}

In this section, we review the definition of the ring A_{crys} and prove some properties (Corollary 2.3 and 2.5) which we need in the next section. See [Fo1], [Fo2], [Fo3].

Let K, k, O_K, $\overline{K}$, and G_K be as in the previous section. Let $O_{\overline{K}}$ be the ring of integers of $\overline{K}$, and define the ring R to be the projective limit of

$$O_{\overline{K}}/pO_{\overline{K}} \overset{\text{frob.}}{\longleftarrow} O_{\overline{K}}/pO_{\overline{K}} \overset{\text{frob.}}{\longleftarrow} O_{\overline{K}}/pO_{\overline{K}} \overset{\text{frob.}}{\longleftarrow} \cdots,$$

where frob. denotes the absolute frobenius of $O_{\overline{K}}/pO_{\overline{K}}$. Consider the ring of Witt-vectors $W(R)$ with coefficients in R. Define the ring homomorphism $\theta\colon W(R) \to O_C$ by

$$\theta(u) = \lim_{m\to\infty}(\widetilde{u_{0,m}}^{p^m} + p\widetilde{u_{1,m-1}}^{p^{m-1}} + \cdots + p^m\widetilde{u_{m,0}}).$$
$$u = (u_0, u_1, u_2, \cdots) \in W(R),\ u_n = (u_{n,0}, u_{n,1} \cdots) \in R,\ u_{n,m} \in O_{\overline{K}}/pO_{\overline{K}}.$$

Here C is the completion of $\overline{K}$, O_C is its ring of integers, and $\tilde{}$ denotes a lifting of an element of $O_{\overline{K}}/pO_{\overline{K}}$ to $O_{\overline{K}}$. The homomorphism θ is surjective.

Put $J = \operatorname{Ker} \theta$. Let $D_J(W(R))$ be the PD-envelope of J compatible with the unique PD-structure on $p\mathbb{Z}_p$, and let $\overline{J}$ be its PD-ideal. Define the ring A_{crys} to be the p-adic completion of $D_J(W(R))$. The homomorphism θ, the frobenius of $W(R)$, and the action of the Galois group G_K extend to $D_J(W(R))$ and A_{crys}.

Choose elements $\nu_n \in O_{\overline{K}}$ $(n \geq 0)$ such that $\nu_0 = -p$ and $\nu_{n+1}^p = \nu_n$ $(n \geq 0)$ and define the elements $\underline{-p} \in R$ and $\xi \in W(R)$ by $\underline{-p} = (\nu_n \bmod p)_{n\geq 0}$ and $\xi = p + [\underline{-p}]$. Here $[\ ]: R \to W(R)$ denotes the Teichmüller character. Then $\overline{J}$ is generated by ξ. The sequences (p, ξ) and (ξ, p) form regular sequences in $W(R)$, from which it follows that $D_J(W(R))$ is isomorphic to the $W(R)$-sub-algebra of $W(R)[1/p]$ generated by $\xi^m/m!$ $(m \geq 1)$ and the r-th divided-power $J^{[r]}$ is the ideal generated by $\xi^m/m!$ $(m \geq r)$.

Define the descending filtration Fil^r $(r \in \mathbb{Z})$ of $W(R)$ and $D_J(W(R))$ by

$$Fil^r W(R) = J^r \quad (\text{if } r \geq 0), \quad W(R) \quad (\text{if } r < 0)$$

and

$$Fil^r D_J(W(R)) = \overline{J}^{[r]} \quad (\text{if } r \geq 0), \quad D_J(W(R)) \quad (\text{if } r < 0).$$

Then we have the following isomorphisms.

$$\mathrm{gr}^0 W(R) \xrightarrow{\sim} \mathrm{gr}^r W(R);\ a \mapsto a \cdot \xi^r \quad (r \geq 0) \tag{2.1}$$

$$\mathrm{gr}^0 W(R) \xrightarrow{\sim} \mathrm{gr}^r D_J(W(R));\ a \mapsto a \cdot \xi^{[r]} \quad (r \geq 0) \tag{2.2}$$

By the isomorphism (2.2), $\mathrm{gr}^r D_J(W(R))$ is flat over $\mathbb{Z}_p$. Hence we can define the filtration of A_{crys} by the p-adic completion of that of $D_J(W(R))$. We have an isomorphism

$$\mathrm{gr}^0 W(R) \xrightarrow{\sim} \mathrm{gr}^r A_{\mathrm{crys}};\ a \mapsto a \cdot \xi^{[r]} \quad (r \geq 0). \tag{2.3}$$

The canonical morphism $W(R) \to A_{\mathrm{crys}}$ is injective. We regard the ring $W(R)$ as a subring of A_{crys}. The ring $W(R)$ is closed in A_{crys} with respect to the p-adic topology of A_{crys} and we have $Fil^r W(R) = Fil^r A_{\mathrm{crys}} \cap W(R)$ $(r \geq 0)$.

Choose elements ε_n $(n \geq 0)$ of $O_{\overline{K}}$ such that $\varepsilon_0 = 1$, $\varepsilon_{n+1}^p = \varepsilon_n$ and $\varepsilon_1 \neq 1$. Put $\varepsilon = (\varepsilon_n \bmod p)_{n \geq 0} \in R$ and $\pi_\varepsilon = [\varepsilon] - 1 \in Fil^1 W(R)$.

Define the filtration $I^{[r]}W(R)$ of $W(R)$ by

$$I^{[r]}W(R) = \{a \in W(R) \mid \varphi^n(a) \in Fil^r W(R) \text{ for all } n \geq 0\}$$

Proposition 2.1 ([Fo3] **5.1.4**). *The ideal $I^{[1]}W(R)$ is generated by π_ε.*

We define the filtration $I^{[r]}A_{\mathrm{crys}}$ in the same way. Then $I^{[r]}A_{\mathrm{crys}}$ $(r \geq 1)$ is a PD-ideal of A_{crys}. Let t be the element of $Fil^1 A_{\mathrm{crys}}$ defined by

$$t = \log([\varepsilon]) = \sum_{m \geq 1} (-1)^{m-1}(m-1)!\pi_\varepsilon^{[m]}$$

We have $\sigma(t) = \chi(\sigma)t$ $(\sigma \in G_K)$, $\varphi(t) = p \cdot t$, and $t^{p-1} \in pA_{\mathrm{crys}}$. Here $\chi : G_K \to \mathbb{Z}_p^*$ is the cyclotomic character. For $n \geq 0$, we define $t^{\{n\}} \in A_{\mathrm{crys}}$ by

$$t^{\{n\}} = t^{r(n)}(t^{p-1}/p)^{[q(n)]},$$

where $n = (p-1)q(n) + r(n)$, $0 \leq r(n) < p-1$.

Proposition 2.2 ([Fo3] **5.3.1**). *We have*

$$I^{[r]}A_{\mathrm{crys}} = \left\{ \sum_{s \geq r} a_s t^{\{s\}} \;\middle|\; a_s \in W(R),\ a_s \text{ } p\text{-adically converges to } 0 \right\}$$

for $r \geq 0$.

Corollary 2.3. *We have the following isomorphism for $r \geq 0$.*

$$W(R)/I^{[1]}W(R) \xrightarrow{\sim} I^{[r]}A_{\mathrm{crys}}/I^{[r+1]}A_{\mathrm{crys}};\ a \mapsto a \cdot t^{\{r\}}.$$

Proof. The surjectivity follows from the proposition and the injectivity follows from the facts that $at^{\{r\}}$ is contained in $Fil^{r+1}A_{\mathrm{crys}}$ if and only if a is contained in $Fil^1 A_{\mathrm{crys}}$ for $a \in A_{\mathrm{crys}}$ and $Fil^1 A_{\mathrm{crys}} \cap W(R) = Fil^1 W(R)$. □

As π_ε is contained in $Fil^1 W(R)$,

$$[\varepsilon]^a = \sum_{m \geq 0} a(a-1)\cdots(a-m+1)\pi_\varepsilon^{[m]} \quad (a \in \mathbb{Z}_p)$$

converges p-adically in A_{crys} and contained in $W(R)$. Define the element q and q' of $W(R)$ by $q = \sum_{\zeta \in \mu_{p-1} \cup \{0\} \subset \mathbb{Z}_p} [\varepsilon]^\zeta$ and $q' = \varphi^{-1}(q)$. The ideal $Fil^1 W(R)$ is generated by q'.

Define the descending filtration $Fil^r_p A_{\mathrm{crys}}$ of A_{crys} by

$$Fil^r_p A_{\mathrm{crys}} = \{a \in Fil^r A_{\mathrm{crys}} |\ \varphi(a) \in p^r A_{\mathrm{crys}}\}.$$

Theorem 2.4 ([Fo3] **5.3.6**). *We have an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p t^{\{r\}} \longrightarrow Fil^r_p A_{\mathrm{crys}} \xrightarrow{1-\frac{\varphi}{p^r}} A_{\mathrm{crys}} \longrightarrow 0$$

for $r \geq 0$.

Corollary 2.5. *Let r, s be non-negative integers. Under the same assumption as Theorem 2.4, we have an isomorphism*

$$\frac{W(R)}{\varphi^{-1}(I^{[1]}W(R))} \xrightarrow{\sim} \frac{I^{[s]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}}{I^{[s+1]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}};\ a \mapsto a \cdot q'^{r-s} t^{\{s\}},$$

if $s < r$, and an isomorphism

$$\frac{W(R)}{I^{[1]}W(R)} \xrightarrow{\sim} \frac{I^{[s]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}}{I^{[s+1]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}} = \frac{I^{[s]}A_{\mathrm{crys}}}{I^{[s+1]}A_{\mathrm{crys}}};\ a \mapsto a \cdot t^{\{s\}}$$

if $s \geq r$.

Proof. The latter follows from Proposition 2.2 and Corollary 2.3. We will prove the former. If $(1-\varphi/p^r)(a)$ is contained in $I^{[s]}A_{\mathrm{crys}}$ for $0 \leq s \leq r$ and $a \in Fil^r_p A_{\mathrm{crys}}$, then $\varphi^m(a) - \varphi^{m-1}(a)/p^r \in Fil^s A_{\mathrm{crys}}$ and $a \in Fil^s A_{\mathrm{crys}}$. By induction on m, $\varphi^m(a) \in Fil^s A_{\mathrm{crys}}$ for all $m \geq 0$, that is, $a \in I^{[s]}A_{\mathrm{crys}}$. Hence by the theorem, we obtain the following exact sequence.

$$0 \longrightarrow \mathbb{Z}_p t^{\{r\}} \longrightarrow I^{[s]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}} \xrightarrow{1-\frac{\varphi}{p^r}} I^{[s]}A_{\mathrm{crys}} \longrightarrow 0$$

Taking a quotient of these exact sequences for s and $s+1$, we obtain the following isomorphism for $s < r$.

$$\frac{I^{[s]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}}{I^{[s+1]}A_{\mathrm{crys}} \cap Fil^r_p A_{\mathrm{crys}}} \xrightarrow[1-\frac{\varphi}{p^r}]{\sim} \frac{I^{[s]}A_{\mathrm{crys}}}{I^{[s+1]}A_{\mathrm{crys}}}.$$

Put $\pi_0 = q - p \in W(R)$. Since $\pi_0 \equiv q'^p = p!q'^{[p]} \equiv 0 \bmod pA_{\text{crys}}$, we have

$$\varphi(q'^{r-s}t^{\{s\}}) = p^r(1+\frac{\pi_0}{p})^{r-s}t^{\{s\}} \in p^r A_{\text{crys}}.$$

Hence $q'^{r-s}t^{\{s\}} \in Fil_p^r A_{\text{crys}} \cap I^{[s]}A_{\text{crys}}$. As $\frac{\pi_0}{p}$ is contained in $I^{[1]}A_{\text{crys}}$ (Consider the image of $\varphi^n(\pi_0)$ under the morphism θ), we have

$$\frac{\varphi}{p^r}(q'^{r-s}t^{\{s\}}) \equiv t^{\{s\}} \bmod I^{[s+1]}A_{\text{crys}}.$$

Therefore we obtain the following commutative diagram. The bottom horizontal arrow is defined by $a \mapsto q'^{r-s}a - \varphi(a)$.

$$\begin{array}{ccc}
\dfrac{I^{[s]}A_{\text{crys}} \cap Fil_p^r A_{\text{crys}}}{I^{[s+1]}A_{\text{crys}} \cap Fil_p^r A_{\text{crys}}} & \xrightarrow[1-\frac{\varphi}{p^r}]{\sim} & \dfrac{I^{[s]}A_{\text{crys}}}{I^{[s+1]}A_{\text{crys}}} \\
{\scriptstyle q'^{r-s}t^{\{s\}}}\big\uparrow & & \big\uparrow{\scriptstyle \wr}\ \text{Corollary 2.3} \\
W(R) & \longrightarrow & W(R)/I^{[1]}W(R)
\end{array}$$

Hence it suffices to prove the following lemma. □

Lemma 2.6. *For $n \geq 1$, $q'^n \cdot \varphi^{-1}(I^{[1]}W(R))$ is contained in $I^{[1]}W(R)$ and the morphism*

$$W(R)/\varphi^{-1}(I^{[1]}W(R)) \longrightarrow W(R)/I^{[1]}W(R);\ a \mapsto q'^n a - \varphi(a)$$

is an isomorphism.

Proof. As $q' \in Fil^1 W(R)$, the first statement is trivial. If $\varphi(a) - q'^n a$ is contained in $I^{[1]}W(R)$ for $a \in W(R)$, $\varphi^m(a) - \varphi^{m-1}(q'^n)\varphi^{m-1}(a)$ is contained in $Fil^1 W(R)$ for all $m \geq 1$. By induction on m, it follows that $\varphi^m(a)$ is contained in $Fil^1 W(R)$ for $m \geq 1$, that is, $\varphi(a) \in I^{[1]}W(R)$. Hence this homomorphism is injective. As for the surjectivity, it is enough to show that the homomorphism $R \to R$; $a \mapsto q'^n a - a^p$ is surjective since $W(R)$ is p-adically complete and separated. This is easy. □

3. Invariance under Tate twist of $\mathcal{H}^q(\mathcal{S}_n^{\sim}(r)_{\overline{X}})$

Let X be a quasi-compact and separated scheme which is smooth over O_K.

Choose $\varepsilon_n \in O_{\overline{K}}$ as in §2. We define an element t of $\Gamma(\overline{Y}, \mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{\overline{X}}))$ as follows. Let L be a finite sub-extension of $\overline{K}/K$ such that $\varepsilon_n \in L$, and let $X_L = X \otimes_{O_K} O_L$. If we are given globally a closed immersion $X_L \hookrightarrow Z_L$ with Z_L smooth over W and a compatible system of liftings of frobenius $\{F_{Z_{L,n}} \colon Z_{L,n} \to Z_{L,n}\}$, then we can define an element t of $\Gamma(Y_L, \mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{X_L,Z_L}))$ by

$$\log(\widetilde{\varepsilon_n}^{p^n} - 1) \in \Gamma(Y_L, J_{D_{L,n}})^{\varphi=p},$$

where $\tilde{}$ denotes a lifting of an element of $\mathcal{O}_{X_{L,n}}$ to $\mathcal{O}_{D_{L,n}}$. For general X, we can glue this element and obtain an element t of $\Gamma(Y_L, \mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{X_L}))$. Define an element t of $\Gamma(\overline{Y}, \mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{\overline{X}}))$ by the image of this element. This is independent of the choice of L.

By the product structure of $\mathcal{S}_n^{\sim}(r)_{\overline{X}}$ (cf. [Ka1] I§2), we can define a map

$$\mathcal{H}^q(\mathcal{S}_n^{\sim}(q)_{\overline{X}}) \longrightarrow \mathcal{H}^q(\mathcal{S}_n^{\sim}(r)_{\overline{X}});\ a \mapsto t^{r-q}\cdot a \tag{3.1}$$

for $0 \le q \le r$.

Theorem 3.1. *For $0 \le q \le r$, there exists a positive integer N which depends only on r and q such that the kernel and the cokernel of the morphism (3.1) are killed by p^N for every $n \ge 1$.*

Remark 3.2. When $0 \le q \le r < p-1$, K. Kato has proved in [Ka1] that

$$\mathcal{H}^q(\mathcal{S}_n(q)_{\overline{X}}) \longrightarrow \mathcal{H}^q(\mathcal{S}_n(r)_{\overline{X}});\ a \mapsto t^{r-q}\cdot a$$

is an isomorphism. See [Ka1] or the beginning of §4 for the definition of $\mathcal{S}_n(r)$.

As the question is etale local on X, we may assume that X is isomorphic to the base change of a smooth scheme over W. Hence we may assume that $O_K = W$. Furthermore we may assume that k is algebraically closed (hence $\overline{Y} = Y$), and there exists a lifting of frobenius $F_X : X \to X$. Choose F_X.

Lemma 3.3 ([Ka1] **I Lemma (4.6)). ***The object $\mathcal{S}_n^{\sim}(r)_{\overline{X}}$ is isomorphic to the mapping fiber of*

$$p^r - \varphi\otimes d\varphi\colon\ Fil^r A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n} \longrightarrow A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n},$$

where $d\varphi$ is the morphism induced by F_X. By this isomorphism, the element $t \in \Gamma(Y,\mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{\overline{X}}))$ corresponds to the element given by $t\otimes 1 \in Fil^1 A_{\mathrm{crys}}\otimes_W \Gamma(Y,\mathcal{O}_{X_n})$

Let $Fil_p^r A_{\mathrm{crys}}$ and $I^{[r]}A_{\mathrm{crys}}$ be as in §2. Let $C_n(r)$ be the mapping fiber of

$$1-\varphi_r\colon\ Fil_p^{r-\cdot} A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n} \longrightarrow A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n},$$

where φ_r is the morphism whose degree q-part is $\frac{\varphi}{p^{r-q}}\otimes\wedge^q\frac{d\varphi}{p}$. Consider the filtration

$$(I^{[s]}A_{\mathrm{crys}}\cap Fil_p^{r-\cdot}A_{\mathrm{crys}})\otimes_W \Omega^{\cdot}_{X_n/W_n} \subset Fil_p^{r-\cdot}A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n}$$

and

$$I^{[s]}A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n} \subset A_{\mathrm{crys}}\otimes_W \Omega^{\cdot}_{X_n/W_n}.$$

Then the morphism φ_r preserves this filtration and we can define a filtration $I^rC_n(r)$ of $C_n(r)$.

Lemma 3.4. *Let r, r' and q be non-negative integers such that $r' \ge r$.*

(1) $\mathcal{H}^q(\mathrm{gr}_I^s C_n(r)) = 0$ if $s \ne r-q$.

(2) If $r-q \ge 0$, the morphism

$$\mathcal{H}^q(\mathrm{gr}_I^{r-q} C_n(r)) \xrightarrow{t^{\{r'-r\}}} \mathcal{H}^q(\mathrm{gr}_I^{r'-q} C_n(r'))$$

factors into

$$\mathcal{H}^q(\mathrm{gr}_I^{r-q} C_n(r)) \xrightarrow{\sim} \mathcal{H}^q(\mathrm{gr}_I^{r'-q} C_n(r')) \xrightarrow{\alpha} \mathcal{H}^q(\mathrm{gr}_I^{r'-q} C_n(r')),$$

where $\alpha \in \mathbb{Z}_p$ is defined by $t^{\{r'-r\}}\cdot t^{\{r-q\}} = \alpha\cdot t^{\{r'-q\}}$.

(3) $\mathcal{H}^q(I^sC_n(r)) = 0$ if $s \ge r-q+2$.

Proof. Put $IW(R) = I^{[1]}W(R)$ and $I'W(R) = \varphi^{-1}(I^{[1]}W(R))$ (§2) to simplify the notation. By Corollary 2.3 and 2.5, we get the following isomorphisms.

$$\frac{W(R)}{IW(R)} \otimes_W \Omega^q_{X_n/W_n} \xrightarrow{\sim} \mathrm{gr}^s_I(A_{\mathrm{crys}} \otimes_{W_\cdot} \Omega^q_{X_n/W_n}) \quad (s \geq 0)$$
$$\omega \mapsto t^{\{s\}} \cdot \omega$$
$$\frac{W(R)}{I'W(R)} \otimes_W \Omega^q_{X_n/W_n} \xrightarrow{\sim} \mathrm{gr}^s_I(Fil^{r-q}_p A_{\mathrm{crys}} \otimes_W \Omega^q_{X_n/W_n}) \quad (0 \leq s < r-q)$$
$$\omega \mapsto q'^{r-q-s} t^{\{s\}} \cdot \omega$$
$$\frac{W(R)}{IW(R)} \otimes_W \Omega^q_{X_n/W_n} \xrightarrow{\sim} \mathrm{gr}^s_I(Fil^{r-q}_p A_{\mathrm{crys}} \otimes_W \Omega^q_{X_n/W_n}) \quad (s \geq r-q)$$
$$\omega \mapsto t^{\{s\}} \cdot \omega$$

Hence the morphism

$$1 - \varphi_r \colon \mathrm{gr}^s_I(Fil^{r-\cdot}_p A_{\mathrm{crys}} \otimes_W \Omega^\cdot_{X_n/W_n}) \longrightarrow \mathrm{gr}^s_I(A_{\mathrm{crys}} \otimes_W \Omega^\cdot_{X_n/W_n})$$

for $s \geq 0$ is described as follows. (See the proof of Corollary 2.5)

$$\begin{array}{ccccccc}
\xrightarrow{pd} & \frac{W(R)}{I'W(R)} \otimes \Omega^{r-s-2} & \xrightarrow{pd} & \frac{W(R)}{I'W(R)} \otimes \Omega^{r-s-1} & \xrightarrow{q'd} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s} \\
& \Big\downarrow{\scriptstyle q'^2 - \varphi \otimes \wedge^{r-s-2} \frac{d\varphi}{p}} & & \Big\downarrow{\scriptstyle q' - \varphi \otimes \wedge^{r-s-1} \frac{d\varphi}{p}} & & \Big\downarrow{\scriptstyle 1 - \varphi \otimes \wedge^{r-s} \frac{d\varphi}{p}} \\
\xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s-2} & \xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s-1} & \xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s} \\
\xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s+1} & \xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s+2} & \xrightarrow{d} & \cdots \\
& \Big\downarrow{\scriptstyle 1 - p\varphi \otimes \wedge^{r-s+1} \frac{d\varphi}{p}} & & \Big\downarrow{\scriptstyle 1 - p^2 \varphi \otimes \wedge^{r-s+2} \frac{d\varphi}{p}} & & \\
\xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s+1} & \xrightarrow{d} & \frac{W(R)}{IW(R)} \otimes \Omega^{r-s+2} & \xrightarrow{d} & \cdots
\end{array} \tag{3.2}$$

(2) easily follows from this.

Proof of (1): We may assume that $s \geq 0$. As we have a short exact sequence

$$0 \longrightarrow \mathrm{gr}^s_I C_n(r) \xrightarrow{p} \mathrm{gr}^s_I C_{n+1}(r) \longrightarrow \mathrm{gr}^s_I C_1(r) \longrightarrow 0,$$

we can reduce to the case $n = 1$ by induction on n. As $\varphi(q') \equiv p \mod Fil^1 W(R)$ and $q' \in Fil^1 W(R)$, an element a of $W(R)$ is contained in $I'W(R)$ if and only if $q'a$ is in $IW(R)$ i.e. the morphism $q' \colon W(R)/I'W(R) \to W(R)/IW(R)$ is injective. As $q' - p \in I'W(R)$, $q'^2 \equiv pq' \mod IW(R)$. On the other hand, we have an isomorphism

$$\begin{aligned} W(R)/I'W(R) \otimes_W \Omega^q_{X_1/k} &\cong {}_\varphi(W(R)/IW(R)) \otimes_W \Omega^q_{X_1/k} \\ &\xrightarrow[C^{-1}]{\sim} \mathcal{H}^q(W(R)/IW(R) \otimes_W \Omega^\cdot_{X_1/k}), \end{aligned} \tag{3.3}$$

which coincides with $\varphi\otimes\wedge^q\frac{d\varphi}{p}$, and it is easy to see that the homomorphism

$$1-\varphi\otimes\wedge^q\frac{d\varphi}{p}\colon Z^q(W(R)/IW(R)\otimes_W\Omega^{\cdot}_{X_1/k})\longrightarrow\mathcal{H}^q(W(R)/IW(R)\otimes_W\Omega^{\cdot}_{X_1/k})$$

is surjective. Hence, from (3.2), we obtain

$$\mathcal{H}^q(\mathrm{gr}^s_I C_1(r))=0$$

if $s\le r-q-2$ or $s\ge r-q+1$ ($q\le r-s-2$ or $q\ge r-s+1$), and

$$\mathcal{H}^q(\mathrm{gr}_I^{r-q-1}C_1(r))\cong\mathrm{Ker}\left\{Z^q(\frac{W(R)}{I'W(R)}\otimes\Omega^{\cdot}_{X_1/k})\overset{q'-\varphi\otimes\wedge^q\frac{d\varphi}{p}}{\longrightarrow}\mathcal{H}^q(\frac{W(R)}{IW(R)}\otimes\Omega^{\cdot}_{X_1/k})\right\}$$

if $s=r-q-1$ ($q=r-s-1$). It remains to prove the injectivity of $q'-\varphi\otimes\wedge^q\frac{d\varphi}{p}$. By $W(R)/pW(R)\cong R$ and Proposition 2.1, the projection to the second component $R\to O_{\overline{K}}/pO_{\overline{K}}$; $(u_n)_{n\ge0}\mapsto u_1$ gives isomorphisms

$$W(R)/IW(R)\otimes\mathbb{Z}/p\mathbb{Z}\cong O_{\overline{K}}/(\varepsilon_1-1)O_{\overline{K}}$$
$$W(R)/I'W(R)\otimes\mathbb{Z}/p\mathbb{Z}\cong O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}.$$

Hence the morphism in problem is described as follows. Here q_1' denotes the image of q' under the above homomorphisms.

(3.4)

$$q_1'-\varphi\otimes\wedge^q\frac{d\varphi}{p}\colon Z^q(O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}\otimes\Omega^{\cdot}_{X_1/k})\to\mathcal{H}^q(O_{\overline{K}}/(\varepsilon_1-1)O_{\overline{K}}\otimes\Omega^{\cdot}_{X_1/k})$$

Define the filtrations of RHS by the images of LHS in the following commutative diagram.

$$\begin{array}{ccc}(\frac{\varepsilon_2-1}{\varepsilon_{n+2}-1}O_{\overline{K}})/(\varepsilon_2-1)O_{\overline{K}}\otimes\Omega^q_{X_1/k} & \longrightarrow & O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}\otimes\Omega^q_{X_1/k}\\ \wr\big\downarrow\varphi\otimes\wedge^q\frac{d\varphi}{p} & & \wr\big\downarrow\varphi\otimes\wedge^q\frac{d\varphi}{p}\\ \mathcal{H}^q((\frac{\varepsilon_1-1}{\varepsilon_{n+1}-1}O_{\overline{K}})/(\varepsilon_1-1)O_{\overline{K}}\otimes\Omega^{\cdot}_{X_1/k}) & \longrightarrow & \mathcal{H}^q(O_{\overline{K}}/(\varepsilon_1-1)O_{\overline{K}}\otimes\Omega^{\cdot}_{X_1/k})\end{array}$$

The bijectivity of the vertical arrows are verified in the same way as (3.3), and the two horizontal arrows are injective. We give on $Z^q(O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}\otimes\Omega^{\cdot}_{X_1/k})$ the filtration induced by that of $O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}\otimes\Omega^q_{X_1/k}$. Since $q_1'O_{\overline{K}}/(\varepsilon_1-1)O_{\overline{K}}=\frac{\varepsilon_1-1}{\varepsilon_2-1}O_{\overline{K}}/(\varepsilon_1-1)O_{\overline{K}}$, the morphism (3.4) preserves these filtrations and its gr^n

$$\begin{aligned}&\mathrm{gr}^n(Z^q(O_{\overline{K}}/(\varepsilon_2-1)O_{\overline{K}}\otimes_W\Omega^{\cdot}_{X_1/k}))\\ &\hookrightarrow(\frac{\varepsilon_2-1}{\varepsilon_{n+2}-1}O_{\overline{K}})/(\frac{\varepsilon_2-1}{\varepsilon_{n+3}-1}O_{\overline{K}})\otimes_W\Omega^q_{X_1/k}\\ &\underset{\sim}{\overset{-\varphi\otimes\wedge^q\frac{d\varphi}{p}}{\longrightarrow}}\mathcal{H}^q((\frac{\varepsilon_1-1}{\varepsilon_{n+1}-1}O_{\overline{K}})/(\frac{\varepsilon_1-1}{\varepsilon_{n+2}-1}O_{\overline{K}})\otimes_W\Omega^{\cdot}_{X_1/k})\end{aligned}$$

is injective. Hence it is enough to show

$$(3.5)\qquad \bigcap_n \frac{\varepsilon_2 - 1}{\varepsilon_n - 1} \cdot (O_{\overline{K}} \otimes_W \mathcal{O}_X) = (\varepsilon_2 - 1)(O_{\overline{K}} \otimes_W \mathcal{O}_X).$$

Since $X \otimes_{O_K} O_L$ is normal for any finite sub-extension L of $\overline{K}/K$, we can show (3.5), using the discrete valuations of primes of height 1 containing p.

Proof of (3): As $I^{[s]}A_{\mathrm{crys}} \cap Fil_p^r A_{\mathrm{crys}} = I^{[s]}A_{\mathrm{crys}}$ and $\varphi(I^{[s]}A_{\mathrm{crys}}) \subset pA_{\mathrm{crys}}$ if $s \geq r+1$, the morphism

$$1 - \varphi_r \colon (I^{[s]}A_{\mathrm{crys}} \cap Fil_p^{r-q} A_{\mathrm{crys}}) \otimes_W \Omega^q_{X_n/W_n} \longrightarrow I^{[s]}A_{\mathrm{crys}} \otimes_W \Omega^q_{X_n/W_n}$$

is an isomorphism if $s \geq r - q + 1$. Hence $\mathcal{H}^q(I^{[s]}C_n(r)) = 0$ if $s \geq r - q + 2$. $\square$

Proof of Theorem 3.1. By Lemma 3.3 and $p^r(Fil^r A_{\mathrm{crys}}/Fil_p^r A_{\mathrm{crys}}) = 0$, it is enough to show that for $0 \leq q \leq r$, there exists an integer N' depending only on r and q such that the kernel and the cokernel of the morphism

$$t^{\{r-q\}} \colon \mathcal{H}^q(C_n(q)) \longrightarrow \mathcal{H}^q(C_n(r))$$

is killed by $p^{N'}$. From Lemma 3.4 (1) and (3), we obtain the isomorphism

$$\mathcal{H}^q(C_n(r)) \cong \mathcal{H}^q(\mathrm{gr}_I^{r-q} C_n(r))$$

for $r \geq q \geq 0$. Hence the claim follows from Lemma 3.4 (2). $\square$

4. Proof of Theorem 1.1 in the good reduction case

In this section X is a smooth scheme over O_K. First assume that we are given globally a closed immersion $X \hookrightarrow Z$ into a smooth scheme Z over W and a compatible system of liftings of frobenius $\{F_{Z_n} \colon Z_n \to Z_n\}$. Further assume that these satisfy the following condition.

(*) There exist $T_1, \cdots, T_d \in \Gamma(Z, \mathcal{O}_Z^*)$ such that dT_i form a basis of $\Omega^1_{Z/W}$ and $F_{Z_n}(T_i) = T_i^p$ for $n \geq 1$.

In this situation, we define the complex $\mathcal{S}_n'(r)_{X,Z}$ for $r \geq 0$, which coincides with $\mathcal{S}_n(r)_{X,Z}$ defined in [Ka1] if $r < p$, as follows. Using the fact that X is syntomic over W, we can verify that the following sequence is exact. (See [Ka1] I Lemma (1.3).)

$$J^{[r]}_{D_{n+m}} \xrightarrow{p^m} J^{[r]}_{D_{n+m}} \xrightarrow{p^n} J^{[r]}_{D_{n+m}} \longrightarrow J^{[r]}_{D_n} \longrightarrow 0.$$

Define the subsheaf $\widetilde{J^{[r]\prime}_{D_n}}$ of $\mathcal{O}_{D_n}$ by

$$\widetilde{J^{[r]\prime}_{D_n}} = \{a \in J^{[r]}_{D_n} \mid \varphi_{D_n}(a) \in p^r \mathcal{O}_{D_n}\}.$$

Here $\varphi_{D_n} \colon \mathcal{O}_{D_n} \to \mathcal{O}_{D_n}$ is the homomorphism induced by F_{Z_n}. Then $\widetilde{J^{[r]\prime}_{D_{n+s}}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ $(s \geq r)$ is independent of s, and we describe this sheaf by $J^{[r]\prime}_{D_n}$. Using the above exact sequence, we can verify that $J^{[r]\prime}_{D_n}$ is flat over W_n. Note that $J^{[r]}_{D_n} = J^{[r]\prime}_{D_n}$ if $r < p$. We can define a φ_{D_n}-linear map $\varphi_r\colon J_{D_n}^{[r]\prime} \to \mathcal{O}_{D_n}$ by the following commutative diagram (cf. [Ka1] I Corollary (1.5)).

$$\begin{array}{ccc}
\widetilde{J_{D_{n+r}}^{[r]\prime}} & \xrightarrow{\varphi} & \mathcal{O}_{D_{n+r}} \\
\big\downarrow & & \big\uparrow{\scriptstyle p^r} \\
J_{D_n}^{[r]\prime} & \xrightarrow{\varphi_r} & \mathcal{O}_{D_n}.
\end{array}$$

From the condition $(*)$, it follows that the subsheaves

$$\widetilde{J_{D_n}^{[r-q]\prime}} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n/W_n} \subset \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n/W_n} \quad (r \in \mathbb{Z})$$

give a sub-complex of $\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{\cdot}_{Z_n/W_n}$. Put

$$J_{D_n}^{[r-\cdot]\prime} \otimes \Omega^{\cdot}_{Z_n/W_n} = (\widetilde{J_{D_{n+r}}^{[r-\cdot]\prime}} \otimes \Omega^{\cdot}_{Z_{n+r}/W_{n+r}}) \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

Define the complex $\mathcal{S}'_n(r)_{X,Z}$ to be the mapping fiber of

$$\iota - \varphi_r\colon J_{D_n}^{[r-\cdot]\prime} \otimes \Omega^{\cdot}_{Z_n/W_n} \to \mathcal{O}_{D_n} \otimes \Omega^{\cdot}_{Z_n/W_n}.$$

Here ι is the morphism induced by the canonical morphism

$$\widetilde{J_{D_{n+r}}^{[r-\cdot]\prime}} \otimes \Omega^{\cdot}_{Z_{n+r}/W_{n+r}} \longrightarrow \mathcal{O}_{D_n} \otimes \Omega^{\cdot}_{Z_n/W_n},$$

and φ_r is the morphism whose degree q-part is

$$\varphi_{r-q} \otimes \wedge^q \frac{d\varphi}{p}\colon J_{D_n}^{[r-q]\prime} \otimes \Omega^q_{Z_n/W_n} \longrightarrow \mathcal{O}_{D_n} \otimes \Omega^q_{Z_n/W_n}.$$

If $r < p$, the complex $\mathcal{S}'_n(r)_{X,Z}$ coincides with $\mathcal{S}(r)_{X,Z}$ defined in [Ka1].

We can define the product structure and the symbol map

$$(i^*j_*\mathcal{O}^*_{X_K})^{\otimes q} \longrightarrow \mathcal{H}^q(\mathcal{S}'_n(q)_{X,Z})$$

in the same way as [Ka1]. Here i (resp. j) is the morphism $Y = X \otimes k \hookrightarrow X$ (resp. $X_K \hookrightarrow X$). For an integer $r \geq 0$, we define $\mathbb{Z}/p^n\mathbb{Z}(r)'$ by $\mathbb{Z}/p^n\mathbb{Z}(r)' := (\frac{1}{p^q q!}\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n\mathbb{Z}$, where q is the maximum integer $\leq r/(p-1)$.

Proposition 4.1. *There exists a canonical morphism*

$$\mathcal{S}'_n(r)_{X,Z} \longrightarrow i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)' \tag{4.1}$$

in $D^+(Y, \mathbb{Z}/p^n\mathbb{Z})$ compatible with the product structures and the symbol maps.

Here the symbol map

$$(i^*j_*\mathcal{O}^*_{X_K})^{\otimes q} \longrightarrow i^*R^qj_*\mathbb{Z}/p^n\mathbb{Z}(q)$$

is defined by the map

$$i^*j_*\mathcal{O}^*_{X_K} \longrightarrow i^*R^1j_*\mathbb{Z}/p^n\mathbb{Z}(1)$$

induced by the Kummer sequence and cup products. We do not give a proof of this proposition here. See [Tsu].

By the same argument as [Ku], we can prove the following theorem, using the result of Bloch-Kato [BK].

Theorem 4.2. *Let q be an integer ≥ 0 and let m be the integer $v_p(r!p^r)$, where r is the maximum integer $\leq q/(p-1)$. Then for any $n > m$, if the primitive p^n-th roots of unity are contained in O_K and $p \geq 3$, the morphism*

$$\mathcal{H}^q(\mathcal{S}'_n(q)_{X,Z}) \longrightarrow i^* R^q j_* \mathbb{Z}/p^n\mathbb{Z}(q)'$$

induced by (4.1) *factors into*

$$\begin{array}{ccc} \mathcal{H}^q(\mathcal{S}'_n(q)_{X,Z}) & \longrightarrow & i^* R^q j_* \mathbb{Z}/p^n\mathbb{Z}(q)' \\ \big\downarrow & & \big\uparrow \\ \mathcal{H}^q(\mathcal{S}'_{n-m}(q)_{X,Z}) & \longrightarrow & i^* R^q j_* \mathbb{Z}/p^{n-m}\mathbb{Z}(q), \end{array}$$

where the left vertical map is surjective and the right one is injective. Furthermore the bottom horizontal morphism is an isomorphism

Remark 4.3. Theorem 4.2 has already been proved when $q \leq p-2$ in [Ku]. See also [Ka1].

We have a map

$$\mathcal{S}^{\sim}_n(r)_{X,Z} \longrightarrow \mathcal{S}'_n(r)_{X,Z} \tag{4.2}$$

defined by the multiplication by p on $J^{[r-\cdot]}_{D_n} \otimes \Omega^{\cdot}$ and the identity on $\mathcal{O}_{D_n} \otimes \Omega^{\cdot}$. Combining this with the morphism (4.1), we obtain

$$\mathcal{S}^{\sim}_n(r)_{X,Z} \longrightarrow i^* R j_* \mathbb{Z}/p^n\mathbb{Z}(r)'. \tag{4.3}$$

Now we consider a general X and do not assume the global existence of $X \hookrightarrow Z$ etc. By "gluing together" the locally defined morphism (4.3) and taking "inductive limit", we can construct a morphism

$$\mathcal{S}^{\sim}_n(r)_{\overline{X}} \longrightarrow \bar{i}^* R \bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)' \tag{4.4}$$

in $D^+(\overline{Y}, \mathbb{Z}/p^n\mathbb{Z})$, where $\bar{i}$ (resp. $\bar{j}$) is the morphism $\overline{Y} \hookrightarrow \overline{X} := X \otimes O_{\overline{K}}$ (resp. $X_{\overline{K}} \hookrightarrow \overline{X}$). If X is proper over O_K, we obtain a morphism

$$H^m(\overline{X}, \mathcal{S}_{\mathbb{Q}_p}(r)) \longrightarrow H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \xrightarrow{1/p^r} H^m_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p(r)), \tag{4.5}$$

compatible with the product structures, where the first morphism is the one induced by (4.4).

Proof of Theorem 1.1. As the morphism (4.2) is a quasi-isomorphism up to bounded torsions, it follows from Theorem 4.2 and Theorem 3.1 that the morphism

$$\mathcal{H}^q(\mathcal{S}^{\sim}_n(r)_{\overline{X}}) \longrightarrow \bar{i}^* R^q \bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)'$$

induced by (4.4) is an isomorphism up to bounded torsions if $r \geq q \geq 0$. Then it follows that the morphism (4.5) is an isomorphism. □

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